Ma 109b, HW #9. Final version. Due Wednesday, March 8.

Note: For the rest of the quarter, I will be covering homology theory. This is discussed in Section 8 of Armstrong, but a better reference is Allen Hatcher's "Algebraic Topology". This is book is also available online at: `http://www.math.cornell.edu/~hatcher/`. Initially, we will be covering the first 20 pages of Chapter 2.
Final: The final will be given out on Wednesday, March 8, and due Thursday, March 16 at 10:00am. It will be a timed exam with a very long time limit, probably 48 hours.

1. In class, I demonstrated the existence of a regular right-angle pentagon (that is, one with all sides the same length). Prove or disprove: This is a unique right-angle pentagon, up to isometry.

   Note: There is a 3-parameter family of right-angle hexagons. More specifically, you can freely specify the lengths of every other side. This fact is a key lemma in computing the dimension of the space of hyperbolic structures on a given surface.

2. Let S be the surface $P\#P\#P$. Prove that S has a hyperbolic structure coming by gluing together some number of regular right-angle pentagons.

   Draw a tiling of $\mathbb{H}^2$ which is symmetric under $\pi_1(S)$ where each tile T is a union of pentagons such that the following is true. Let $p\colon \mathbb{H}^2 \to S$ be the covering map. Then p restricted to T is surjective and p restricted to $\text{int}(T)$ is injective. (Such a T is called a *fundamental domain.*)

   Note: You can just draw said tiling on top of the tiling by regular right angle pentagons handed out in class. If you have misplaced your copy, there is one as part of the online lecture notes for Feb. 27.

3. Compute the homology of S^2 and S^3 directly from the definition. For S^2, think of it as the simplicial complex which is the boundary of a 3-simplex (so you will have four 2-simplices). For S^3 you can use any simplicial complex K with $|K| = S^3$. If you want, you can use Hatcher's weaker notion of a Δ-complex which allows you to get away with fewer simplices.

4. Compute the homology of the projective plane and the Klein bottle.

5. Let F be a field. Suppose you have a finite chain-complex of finite-dimensional vector spaces:
   $$0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_1} C_0 \to 0$$
   In particular, $\partial_n \circ \partial_{n+1} = 0$. Consider the homology $H_n = \ker\partial_n \big/ \operatorname{im}\partial_{n+1}$. Prove that
   $$\sum_i (-1)^i \dim H_i = \sum (-1)^i \dim C_i$$

   Note: This problem may seem highly peculiar at the moment, but it will play a key role in the proof that if K_1 and K_2 are triangulations of the same topological space X then $\chi(K_1) = \chi(K_2)$, just like you showed when X is a surface on HW #1.