Ma 109b, HW #7. Due Wednesday, February 22.

1. Let p be a point in a smooth surface S. Let e_1, e_2 be an orthonormal basis for T_pS and consider the geodesic polar coordinates

$$f\colon (0,R_0)\times(0,2\pi)\to S \quad \text{given by } f(r,\theta)=\exp_p(r\cos\theta e_1 + r\sin\theta e_2)$$

   Prove the following identities which were used in class

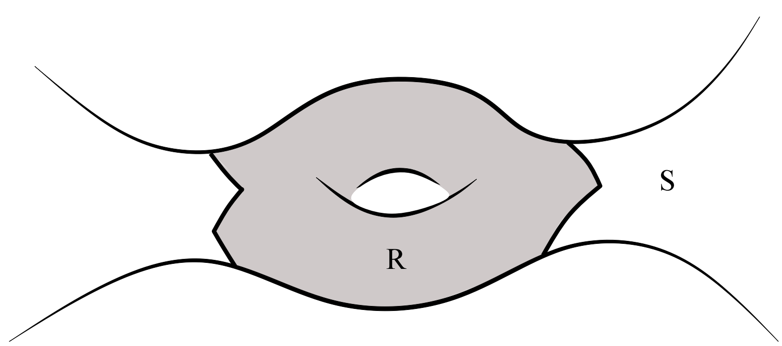
$$\begin{aligned} f_{rr} &= & -L_{11}n \\ f_{\theta r} &= \tfrac{G_r}{2G}f_\theta & -L_{21}n \end{aligned}$$

   where G is the metric coefficient g_{22}, n is the *unit* normal vector in direction $f_r \times f_\theta$, and (L_{ij}) is matrix of the Weingarten map $L = D\hat{n}$ w.r.t. f_r, f_θ.

   Also, derive the corresponding formula for $f_{\theta\theta}$.

2. Let S be a smooth surface in $\mathbb{R}^3$. Supposing that S is compact, prove that there is a point p in S where $K(p) > 0$.

3. Let S be a smooth surface in $\mathbb{R}^3$. Let R be a compact region of S bounded by a family of broken geodesics. Here is an example:



   Derive a version of Gauss-Bonnet for this setting, i.e. express $\int_R K\,dA$ in terms of $\chi(R)$ and the angles between segments of the broken geodesics. You may assume anything that seems reasonable about the existence geodesic triangulations.

4. Let S be a smooth 2-sphere in $\mathbb{R}^3$ where $K > 0$ everywhere. A *closed geodesic* is a geodesic which is an embedded circle. Prove that any two closed geodesics in S must intersect. Is this still true if we drop the requirement on K?

5. The unit sphere has constant curvature $K = 1$, the plane constant curvature $K = 0$. Find a surface in $\mathbb{R}^3$ with constant curvature $K = -1$. Hint: Surfaces of revolution should be your friends, and note that (2) above means that your surface can't be compact.