Ma 109b, HW #5. Due Wednesday, February 15.

1. Prove Lemma 5.6.8 in the text.

2. Let $S \subset \mathbb{R}^3$ be a smooth surface. A *symmetry* of S is an isometry $\phi\colon\ S \to S$.

   (a) Suppose ϕ is a symmetry of S which fixes a point p in S as well as a tangent vector $v \in T_pS$. Prove the geodesic through p with tangent vector v is pointwise fixed by ϕ.

   (b) Use part (a) to show that the geodesics on the round sphere S^2 are exactly the great circles (i.e. the intersections of S^2 with planes through the origin). Also, use it to find some geodesics on a interesting surface of your choice.

3. Let $S \subset \mathbb{R}^3$ be a smooth surface. Recall that the *intrinsic metric* on S is given by

$$d(p,q) = \inf\left\{\text{Length}(c) \;\middle|\; c \text{ as smooth curve joining } p \text{ to } q\right\}$$

   (a) Prove that d is really a metric, that is, check the axioms for a metric given in, for instance, Armstrong §2.4.

   (b) Prove that the topology induced by d is the same as the one S inherits as a subspace of $\mathbb{R}^3$.

   (c) Suppose further that S is closed in $\mathbb{R}^3$. If A is a closed subset of S which is bounded with respect to d, prove that A is compact.

4. Let $S \subset \mathbb{R}^3$ be a *closed* subset which is a smooth surface. The goal of this problem will be to show that for all $p, q \in S$, there exists a geodesic c joining p to q with $\text{Length}(c) = d(p,q)$.

   The key tool is the follow concept. A *broken geodesic* in S is a piecewise smooth curve c consisting of a finite number of geodesic segments glued end to end. Despite the name, a geodesic is an example of a broken geodesic.

   (a) If c is a smooth curve joining p to q, prove there is a broken geodesic $\tilde{c}$ joining p to q with $\text{Length}(\tilde{c}) \le \text{Length}(c)$.

   (b) Suppose that c is a broken geodesic joining p to q. Prove that if c is not a geodesic, then there exists a smooth curve $\tilde{c}$ joining p to q such that $\text{Length}(\tilde{c}) < \text{Length}(c)$.

   (c) Prove there exists a broken geodesic joining p to q whose length is equal to $d(p,q)$.

   (d) Show that there exists a geodesic c joining p to q with $\text{Length}(c) = d(p,q)$.