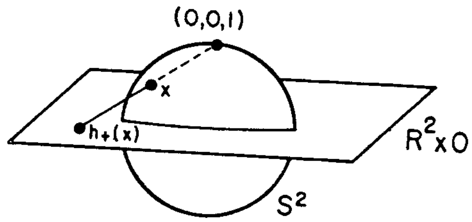
Ma 109b, HW 3. Due Wednesday, January 25

1. Let $S^2 = \{x \in \mathbb{R}^3 \ : \ |x| = 1\}$ be the round unit sphere. Identify $\mathbb{R}^2$ with the (x, y)-plane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^3$. Consider the stereographic projection from the north pole $n = (0,0,1)$,

$$h_+ \colon S^2 \setminus \{n\} \to \mathbb{R}^2,$$

as described in the figure below:



Symmetrically, let h_- be the stereographic projection from the south pole $s = (0,0,-1)$.

(a) Show that h_+^{-1} and h_-^{-1} are coordinate patches for S^2. (As with last week, use my definition of coordinate patch, not the one in the text.)

(b) Compute the change of coordinate map $h_+ \circ h_-^{-1}$, noting the domain and range.

(c) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ be a polynomial with complex coefficients. We will regard p as a smooth function from $\mathbb{R}^2$ to itself by identifying $\mathbb{R}^2$ with $\mathbb{C}$. Define a function $P \colon S^2 \to S^2$ by $P = h_+^{-1} \circ p \circ h_+$ on $S^2 \setminus n$, and $P(n) = n$. Prove that P is smooth.

Note: You should think of n as corresponding to ∞ in some reasonable way. The reason for this subproblem will become clear in Problem 5.

2. Two regular surfaces S_1 and S_2 are *transverse* if for each point p in $S_1 \cap S_2$ we have $T_pS_1 \neq T_pS_2$. Prove that if S_1 and S_2 are transverse, then $S_1 \cap S_2$ is a regular curve.

3. Suppose $\phi \colon S_1 \to S_2$ is a smooth map of smooth surfaces in $\mathbb{R}^3$. For a point $p \in S_1$, we will define the derivative

$$D_p\phi \colon T_pS_1 \to T_{f(p)}S_2$$

of ϕ at p as follows. Given a smooth curve $\alpha \colon (-\epsilon, \epsilon) \to S_1$ with $\alpha(0) = p$, set

$$D_p\phi(\alpha'(0)) = (\phi \circ \alpha)'(0)$$

(a) Prove that $D_p\phi$ is well-defined and is a linear map.

(b) If ϕ is a diffeomorphism, prove that $D_p\phi$ is an isomorphism for all p.

4. Suppose that $\phi \colon S_1 \to S_2$ is a smooth map of surfaces. A point $p \in S_1$ is a *critical point* if $D_p\phi$ is non-invertible. A point $q \in S_2$ is a *critical value* if some point in $f^{-1}(q)$ is a critical point. The complement of the critical values in S_2 are called *regular values.*

(a) Suppose S_1 is compact. Prove that if $q \in S_2$ is a regular value, then $f^{-1}(q)$ is finite.

(b) Again suppose S_1 is compact. Suppose $U \subset S_2$ is a connected set of regular values. Show that $\# f^{-1}(q)$ is constant on U.

(c) Sketch an example to show that (b) fails if U is not connected.

5. Use Problems 1(c) and 4 to prove

   The Fundamental Theorem of Algebra *Let P be a non-constant polynomial with coefficients in $\mathbb{C}$. Then P has a root in $\mathbb{C}$.*

6. Book problem 4.5.1. Do any 2 of the 4 parts.