Math 194c: Topology and Geometry of 3-manifolds

Introduction

- This class is about compact 3-manifolds, for example: S^3, $T^3 = S^1 \times S^1 \times S^1$, $S^2 \times S^1$, or the unit tangent bundle to a surface.

  Fundamental Goal: Classify all compact 3-manifolds. What does classify mean? The ideal classification is that of surfaces:

  Theorem: Every compact connected 2-manifold without boundary is homeomorphic (or diffeomorphic) to one of the following:

  - (Orientable) The sphere, the torus or the connected sum of tori.
  - (Nonorientable) The projective plane, or a connected sum of proj. planes.

  The homeomorphism type of a surface is completely determined by its orientablity and its Euler characteristic (in other words, its homology) which are easily computable from, say, a triangulation.

  The classification of 3-manifolds may or may not be possible (though it probably is). This contrasts with dimensions ≥ 4 where classification of manifolds is impossible.

- The reason classification is impossible in high dimensions is group theoretic. Finitely presented groups can't be classified in any reasonable sense, and for any fixed $n \geq 4$ any finitely presented group is π_1 of some n-manifold. (Proof: For a f.p. group G build a finite 2-complex K with $\pi_1(K) = G$. Now as $n+1 \geq 5$, can embed K in $\mathbb{R}^{n+1}$. Let M be the boundary of a regular nbhd of K. Then M is a closed n-manifold and, since $n \geq 4$, $\pi_1 M = \pi_1 K = G$.) A reason finitely presented groups can't be classified is that there is no algorithm which can decide if two finitely presented groups are isomorphic. In fact, there is no algorithm to decide if a finitely presented group has any of the following properties: trivial, finite, free, nilpotent or simple.

- This doesn't mean you can't say anything about high-dimensional manifolds—in fact high-dimensional topology ($n \geq 5$) is arguably better understood than low-dimensional topology ($n = 3, 4$), once you mod out by the fact that it's impossible. In other words, fix some homotopy type K to get rid of the group theory and look at

  $$\left\{(M, f) \;\middle|\; M \text{ is an } n\text{-manifold, } f\colon M \to K \text{ is a homotopy equivalence.}\right\}.$$

  moded out by homeomorphism (or if you're studying smooth manifolds, diffeomorphism). Often this set can be calculated with homotopy-theoretic methods (stable homotopy groups of spheres, L-groups, surgery exact sequences...).

- Until the 2003 work of Perelman, the following basic question in 3-dimensions was unknown:

  Poincaré conjecture: Let M be a compact 3-manifold without boundary with $\pi_1 M$ trivial. Then M is homeomorphic to S^3.

For a 3-manifold, $\pi_1 M$ trivial is equivalent to M homotopy equivalent to S^3. So you have the generalization:

Gen. Poincaré conjecture: Let M be a compact n-manifold without boundary homotopy equivalent to S^n. Then M is homeomorphic to S^n.

In 1960 Smale proved this was true in dimensions $n \geq 5$. If you replace homeomorphism by diffeomorphism the Generalized Poincaré Conjecture becomes false. For instance, Milnor showed that S^7 has 28 distinct differentiable structures (in general, you can calculate the number of smooth structures on S^n using stable homotopy groups of spheres). In dim 4, Freedman proved the gen. Poincaré around 1980. This illustrates how topology differs as we change dimensions.

- **Geometry:**

  Every surface has a metric of constant curvature. Often these metrics are useful for solving purely topological problems. As a toy example, let Σ be a surface of genus ≥ 2 with some fixed hyperbolic metric. Any homotopy class of simple closed curves in Σ contains a unique geodesic. If we want to study homotopy classes of curves, it is convenient to look at the geodesic representatives since any two geodesics loops:

  - either the same or meet transversely.
  - meet in a minimal number of points (for their homotopy classes).

  Here's a couple of group-theoretic statements about $G = \pi_1(\Sigma)$ whose proofs use the fact Σ has a hyperbolic metric:

  1. G is residually finite, that is, the intersection of all its finite-index subgroups is the identity subgroup.
  2. G is subgroup separable, aka LERF. This means that given a finitely generated subgroup H of G and an element $g \in G - H$ there exists a *finite-index* subgroup H' containing H with $g \notin H'$. The proof works by building the surface out of right-angled pentagons and looking at the induced tiling of $\mathbb{H}^2$.

- It would be nice if all manifolds had metrics of constant curvature, but in higher dimensions, very few manifolds do. One reason for this is that any n-manifold M with a constant curvature metric is a quotient of one of S^n, $\mathbb{E}^n$ or $\mathbb{H}^n$ by a group of isometries. So, for example, $\pi_2(M) = 0$ and hence e.g. $S^2 \times S^2$ or $\mathbb{C}P^2$ don't have such metrics. Also, because $\pi_1(M)$ is a lattice in a Lie group, $\pi_1(M)$ has solvable word problem. However, many finitely presented groups do not have solvable word problem.

  Could generalize constant curvature to locally homogeneous metrics, but still have solvable word problem.

  Around 1980 the theory of 3-manifolds was revolutionized by Thurston's realization that most 3-manifolds *should* have locally homogeneous metrics:

  Geometrization Conjecture: Any compact 3-manifold can be cut into pieces along spheres and tori so that each piece can be given one of the 8 geometric structures: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol, $\widetilde{\mathrm{SL}_2\mathbb{R}}$.

As in dimension 2, the generic case is $\mathbb{H}^3$. If true, this conjecture would be a big step toward classifying 3-manifolds. For instance, it implies that any 3-manifold group has solvable word problem, and is residually finite. In 2003, Perelman announced a proof of the Geometrization Conjecture using Hamilton's approach of Ricci flow.

- In dimension 2 a surface other than S^2 or $\mathbb{R}P^2$ has many constant curvature metrics. It is easy to see that the torus has a 2-dimensional space of flat metrics. For a surface of genus $g \geq 2$, the dimension of the space of hyperbolic metrics, up to isometry, is $6g - 6$. In dimension 3, the same flexibility is true for some geometries like $\mathbb{E}^3$ but in the generic case of $\mathbb{H}^3$ we have:

  Mostow Rigidity: Let M, N be compact hyperbolic n-manifolds with $n \geq 3$. Then if $\pi_1(M)$ is isomorphic to $\pi_1(N)$ then M and N are isometric.

  So for a hyperbolic 3-manifold, geometric invariants such as volume, length of shortest geodesic, or eigenvalues of the Lapacian are actually topological invariants. Dimension 3 is the unique dimension where topology and geometry more or less coincide. Understanding the detailed connections between topology and geometry can be subtle, though, and is one of important areas in the field. Also, there seem to be certain questions which are purely topological that the geometric point of view does not offer much insight. Fortunately, in some of these cases, gauge-theory invariants (especially Floer homology in its many flavors) are powerful tools.

Outline

- **Topological Foundations:** Weeks 1-4. Follows Hatcher and Thurston.
  - Fundamental Goal: Classify compact 3-manifolds.
  - Examples: Triangulations, Heegaard splittings, and Dehn surgery.
  - Categories: Smooth, PL, and Top.
  - Connected sum decomposition.
    - Definitions and examples. Statement of decomposition theorem.
    - Every smooth S^2 in $\mathbb{R}^3$ bounds a ball.
    - Combinatorial minimal surfaces (aka normal surfaces).
    - Proof of theorem.
    - How this allows us to avoid the Poincaré conjecture, much of the time.
  - Homotopy to geometry (more normal surfaces)
    - Loop Theorem.
      - Incompressible surfaces.
    - Sphere Theorem: If M is a 3-manifold and $\pi_2(M) \neq 0$ then there is an *embedded* 2-sphere which is non-trivial in $\pi_2(M)$.
    - Consequences: Many 3-manifolds are $K(\pi, 1)$'s.
  - Normal surfaces and Algorithms for 3-manifolds.

- **The Geometry of 3-manifolds:** Weeks 4-6. Follows Thurston, Bonahon, Scott.
  - Overview, dimension 2, dimensions > 3. Why dimension 3 is so special.
  - The eight 3-dimensional geometries.
  - Seifert fibered spaces.
  - Hyperbolic 3-manifolds.
  - JSJ decomposition theorem (decomposition along tori).
    - Special case of knot complements in S^3.
  - Thurston's Geometrization Conjecture.
  - Consequences of having a geometric structure: properties of the fundamental group.
- **A more recent developement:** Weeks 6-9.