Math 157a Homework #7; Due Monday, February 23

Reminder: The course paper is due at the beginning of class on Wednesday, March 10. This is to be a 6–10 page paper about a topic in Riemannian geometry that interests you. Just to be clear, "paper" implies typed, rather than, handwritten in this context; however, if necessary to avoid excessive inconvenience you can write in some of the symbols or equations by hand. Also, in terms of the requested page length, I'm thinking of a "page of text" as something like one page from our textbook. I would be happy to look at drafts of your paper if you have any concerns about content or form.

Because the paper is due only 3 weeks from now, this HW is shorter than usual. There will be only one more HW assignment after this (#8 due Monday, March 1), and this too will be shorter than average.

1. Prove Scholium 3.78 from GHL:

   Let M be a complete Riemannian manifold, and p a point in M. Show that $q \in M$ is in the cut-locus of p if and only if at least one of the following holds:

   (a) There exist distinct minimal geodesics joining p to q.

   (b) There is a minimal geodesic joining p to q along which p and q are conjugate.

2. Let M be a complete Riemannian manifold with non-positive sectional curvature. Consider points p and q in M. Prove that there is a unique geodesic in each homotopy class of paths joining p to q.

3. Let M be a complete Riemannian manifold with non-positive sectional curvature. Show that a non-trivial element of $\pi_1(M)$ has infinite order.

4. As in past HWs, say that a Riemannian manifold (M, g) is *algebraically locally symmetric* if $DR = 0$ everywhere. A Riemannian manifold (M, g) is *geometrically locally symmetric* if for each p in M there is a small embedded ball $B_p(\epsilon)$ so that map $\exp(v) \mapsto \exp(-v)$ is an isometry on $B_p(\epsilon)$.

   Prove that these two conditions are equivalent.