Math 157a Homework #5; Due Monday, February 9

1. Let (M, g) be a Riemannian manifold, and R the curvature tensor. Show that for all tangent vectors a, b, c, d in T_pM we have $R(a, b, c, d) = R(c, d, a, b)$. (This is the omitted part (iii) of Prop. 3.5 of GHL.)

2. Let S^n be the n-sphere with its standard round metric. Let $\mathrm{Isom}(S^n)$ be the group of Riemannian isometries of S^n. Show that $\mathrm{Isom}(S^n) = \mathrm{O}(n+1)$.

3. Let $\mathscr{H}$ be the Heisenberg group of Ex. 2.90 bis, with the given left-invariant metric. The left action of $\mathscr{H}$ on $\mathscr{H}$ gives an inclusion $\mathscr{H} \to \mathrm{Isom}(\mathscr{H})$. Show that this inclusion is not surjective, and moreover that the dimension of $\mathrm{Isom}(\mathscr{H})$ is at least 4. Note: Using these additional symmetries can greatly simplify the problem 2.90 bis c) from prior problem sets.

4. A closed submanifold N of (M, g) is called *totally geodesic* if every minimal geodesic with endpoints in N is contained in N. That is, N is convex in the sense of the last HW.

   (a) Let N is totally geodesic submanifold of M, and p a point in N. Suppose that P is a plane in $T_p(N)$. Show that $K_N(P) = K_M(P)$, where the former is the sectional curvature measured in N (with the metric inherited from M), and the latter is the sectional curvature measured in M.

   (b) Show that if N is not totally geodesic then the conclusion of part (a) need not hold.

   (c) Suppose N is a closed submanifold of M which satisfies the conclusion of part (a) above. Does N have to be totally geodesic?

5. Let (M, g) be a compact Riemannian manifold, and X and Y be closed submanifolds. Set

$$D = \inf\{d_g(x, y) \mid x \in X \quad \text{and} \quad y \in Y\}.$$

   Prove that D is realized by a geodesic γ with one endpoint in x and the other in y. Show that, moreover, any such geodesic meets X and Y in right angles.

6. As explained in GHL §2.58, a connection on TM extends to a connection on the space of tensors on M. Thus if R is the curvature tensor of type (0,4), given a vector field X we can talk about its covariant derivative D_XR which is also a (0,4) tensor. If we think of X as one of the inputs, then DR is a (0,5) tensor.

   Say that a Riemannian manifold (M, g) is *algebraically locally symmetric* if $DR = 0$ everywhere. A Riemannian manifold (M, g) is *geometrically locally symmetric* if for each p in M there is a small ball $B_p(\epsilon)$ so that map $\exp(v) \mapsto \exp(-v)$ is an isometry on $B_p(\epsilon)$. In later homeworks, you will show that these two conditions are equivalent; the class of Riemannian manifolds satisfying these conditions are called *locally symmetric*. This is one of the most important classes of Riemannian manifolds: It includes manifolds of constant curvature, $\mathbb{C}\mathrm{P}^n$, and compact Lie groups with biinvariant metrics. All of these examples just given are locally homogenous, and this is true of locally symmetric spaces in general.

   (a) Let (M, g) be algebraically locally symmetric. Let c be a geodesic in M. Let X, Y, Z be parallel vector fields along c. Prove that $R(X, Y)Z$ is also parallel.

   (b) Suppose M be a connected Riemannian 2-manifold which is algebraically locally symmetric. Prove that M has constant curvature. (The converse is true in any dimension.)