Annoying trailers:

SnapPy http://snappy.computop.org

What is SnapPy?

SnapPy is a user interface to the SnapPea kernel which runs on Mac OS X, Linux, and Windows. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. You can see it in action, learn how to install it, and read the tutorial.

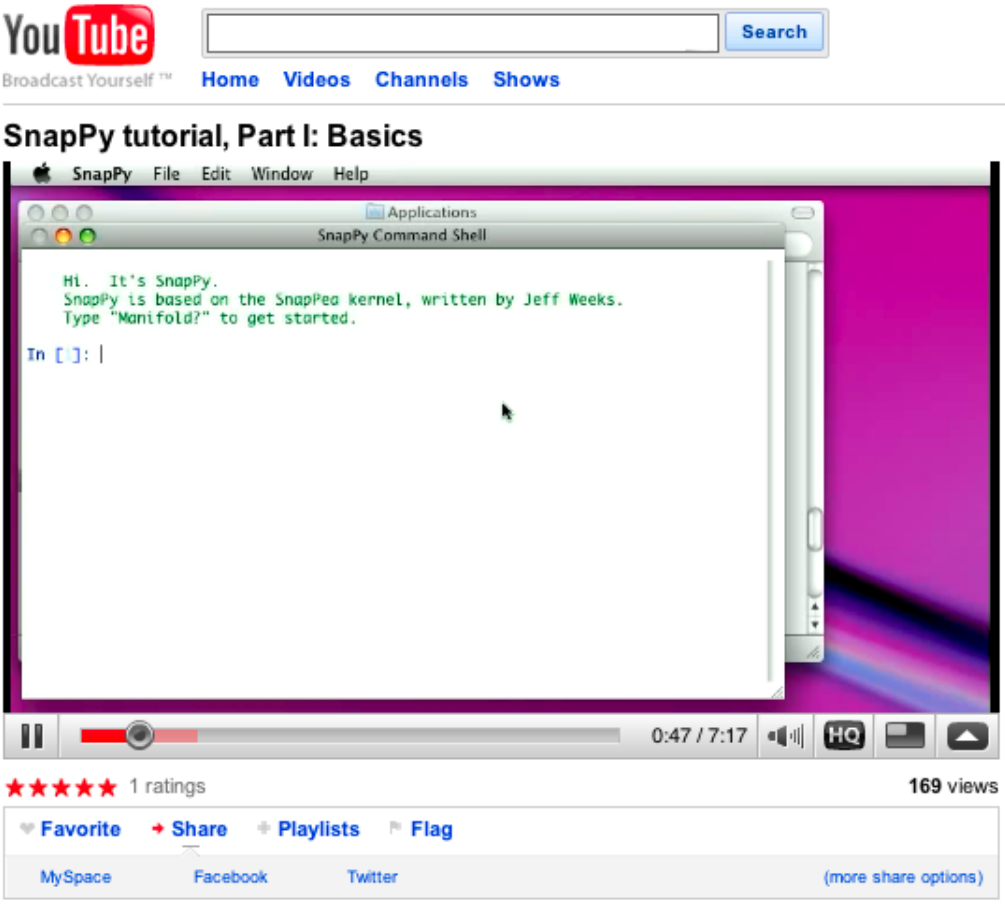
Contents

- Screenshots: SnapPy in action
- Installing and running SnapPy
- Tutorial
- snappy: A Python interface for SnapPea
- Using SnapPy's link editor
- To Do List
- Development Basics: OS X
- Development Basics: Windows XP

Credits

Written by Marc Culler and Nathan Dunfield. Uses the SnapPea kernel written by Jeff Weeks. Released under the terms of the GNU General Public License.

http://www.youtube.com/user/NathanDunfield/



People who heard this talk also viewed:

- J. Aaber and N. Dunfield
  Closed surface bundles of least volume
  arXiv:1002.3423

Hyperbolically twisted Alexander polynomials of knots

Nathan M. Dunfield
University of Illinois

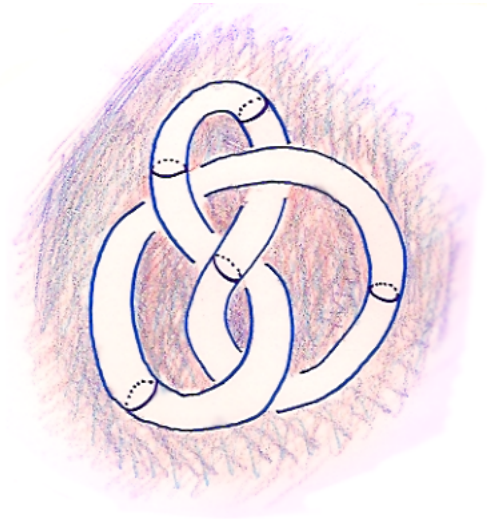
Stefan Friedl
Nicholas Jackson
Warwick

Jacofest, June 4, 2010

This talk available at http://dunfield.info/
Math blog: http://ldtopology.wordpress.com/

Setup:

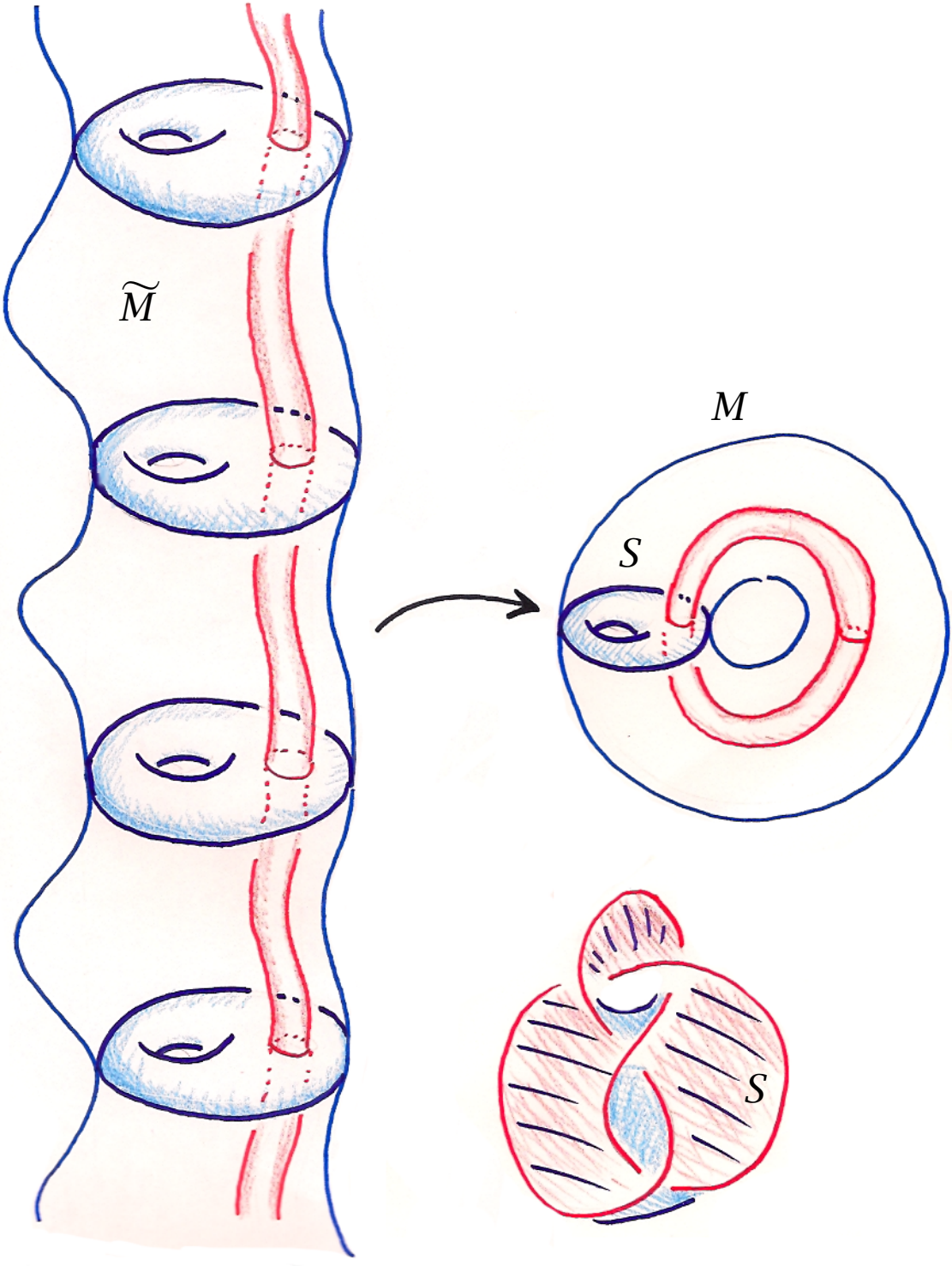
- Knot: $K = S^1 \hookrightarrow S^3$
- Exterior: $M = S^3 - \mathring{N}(K)$



A basic and fundamental invariant of K its *Alexander polynomial* (1923):

$$\Delta_K(t) = \Delta_M(t) \in \mathbb{Z}[t, t^{-1}]$$

Universal cyclic cover: corresponds to the kernel of the unique epimorphism $\pi_1(M) \to \mathbb{Z}$.



$A_M = H_1(\widetilde{M};\mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^{\pm 1}]$, where $\langle t \rangle$ is the covering group.

As Λ is a PID,

$$A_M = \prod_{k=0}^{n} \Lambda \Big/ (p_k(t))$$

Define

$$\Delta_M(t) = \prod_{k=0}^{n} p_k(t) \in \mathbb{Q}[t, t^{-1}]$$

Figure-8 knot:

$$\Delta_M = t - 3 + t^{-1}$$

Genus:

$$\begin{aligned} g &= \min(\text{genus of } S \text{ with } \partial S = K) \\ &= \min(\text{genus of } S \text{ gen. } H_2(M, \partial M; \mathbb{Z})) \end{aligned}$$

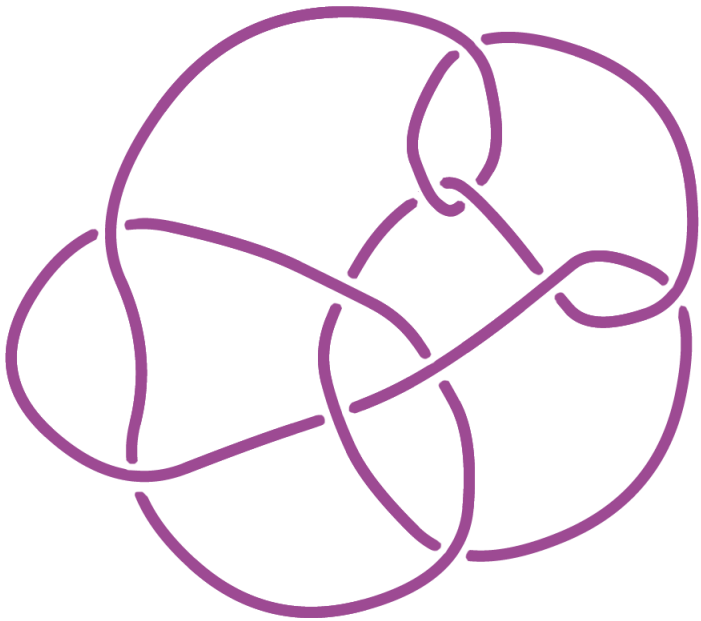
Fundamental fact:

$$2g \geq \deg(\Delta_M)$$

Proof: Note $\deg(\Delta_M) = \dim_{\mathbb{Q}}(A_M)$. As A_M is generated by $H_1(S;\mathbb{Q}) \cong \mathbb{Q}^{2g}$, the inequality follows.

$\Delta(t)$ determines g for all alternating knots and all fibered knots.

Kinoshita-Terasaka knot: $\Delta(t) = 1$ but $g = 2$.



Focus: Improve Δ_M by looking at $H_1(\widetilde{M}; V)$ for some system V of local coefficients.

Assumption: M is hyperbolic, i.e.

$$\mathring{M} = \mathbb{H}^3 \Big/ \Gamma \quad \text{for a lattice } \Gamma \leq \mathrm{Isom}^+ \mathbb{H}^3$$

Thus have a faithful representation

$$\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C} \leq \mathrm{Aut}(V) \quad \text{where } V = \mathbb{C}^2.$$

Hyperbolic Alexander polynomial:

$$\tau_M(t) \in \mathbb{C}\left[t^{\pm 1}\right] \quad \text{coming from } H_1(\widetilde{M}; V_\alpha).$$

Examples:

- Figure-8: $\tau_M = t - 4 + t^{-1}$
- Kinoshita-Terasaka:

$$\begin{aligned}\tau_M \approx &(4.417926 + 0.376029i)(t^3 + t^{-3}) \\ &- (22.941644 + 4.845091i)(t^2 + t^{-2}) \\ &+ (61.964430 + 24.097441i)(t + t^{-1}) \\ &- (-82.695420 + 43.485388i)\end{aligned}$$

Really best to define $\tau_M(t)$ as torsion, a la Reidemeister/Milnor/Turaev.

Basic Properties:

- Can be normalized so $\tau_M(t) = \tau_M(t^{-1})$.
- Then τ_M is an actual element of $\mathbb{C}[t^{\pm 1}]$, in fact of $\mathbb{Q}(\operatorname{tr}(\Gamma))[t^{\pm 1}]$.
- $\tau_{\overline{M}} = \overline{\tau_M(t)}$
- M amphichiral $\Rightarrow \tau_M(t) \in \mathbb{R}[t^{\pm 1}]$.
- $\tau_M(\zeta) \neq 0$ for any root of unity ζ.
- Genus bound:

$$4g - 2 \geq \deg \tau_M(t)$$

For the KT knot, $g = 2$ and $\deg \tau_M(t) = 3$ so this is sharp, unlike with Δ_M.

Knots by the numbers:

313,231	number of prime knots with at most 15 crossings. [HTW 98]
8,834	number where $2g > \deg(\Delta_M)$.
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0	number where $4g - 2 > \deg(\tau_M)$.

Conj. *τ_M determines the genus for any hyperbolic knot in S^3.*

Computing τ_M: Approximate $\pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ to 250 digits by solving the gluing equations associated to some ideal triangulation of M to high precision.

Many properties of M^3 are algorithmically computable, including

[Haken 1961] Whether a knot K in S^3 is unknotted. More generally, can find the genus of K.

[Jaco-Oertel 1984] Whether M contains an incompressible surface.

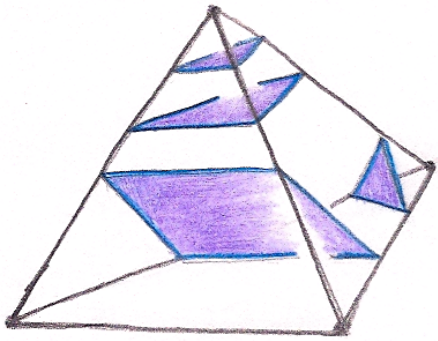
[Rubinstein-Thompson 1995] Whether M is S^3.

[Haken-Hemion-Matveev] Whether two Haken 3-manifolds are homeomorphic.

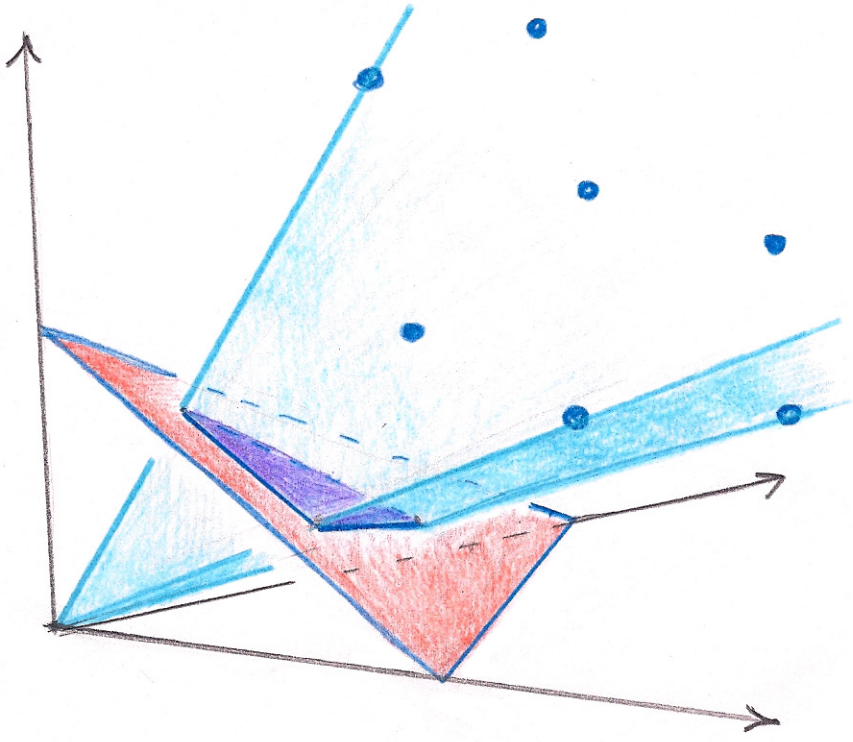
All of these plus Perelman, Thurston, Casson-Manning, Epstein et. al., Hodgson-Weeks, and others give:

Thm. *There is an algorithm to determine if two compact 3-manifolds are homeomorphic.*

Normal surfaces meet each tetrahedra in a triangulation $\mathcal{T}$ of M in a standard way:



and correspond to certain lattice points in a finite polyhedral cone in $\mathbb{R}^{7t}$ where $t = \#\mathcal{T}$:



Meta Thm. *In an interesting class of surfaces, there is one which is normal. Moreover, one lies on a vertex ray of the cone.*

E.g. The class of minimal genus surfaces whose boundary is a given knot.

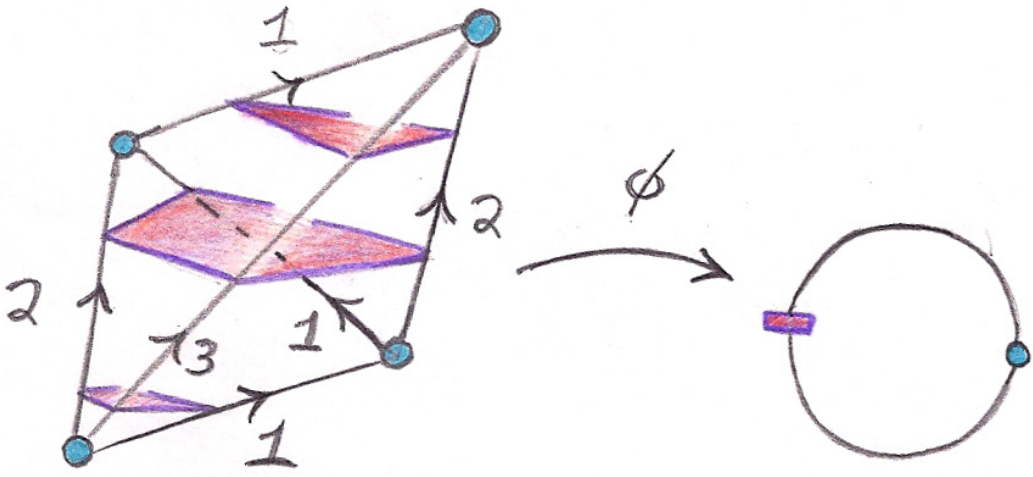
Problem: There can be exponentially many vertex rays, typically $\approx O(1.6^t)$ [Burton 2009]. In practice, limited to $t < 40$.

[Agol-Hass-Thurston 2002] Whether the genus of a knot $K \subset M^3$ is $\leq g$ is NP-complete.

[Agol 2002] When $M = S^3$ the previous question is in co-NP.

Practical Trick: Finding the simplest surface representing some $\phi \in H^1(M;\mathbb{Z}) \cong H_2(M, \partial M;\mathbb{Z})$.

Take a triangulation with only one vertex (cf. Jaco-Rubinstein, Casson). Then ϕ comes from a unique 1-cocycle, which realizes ϕ as a piecewise affine map $M \to S^1$.



Power of randomization: Trying several different $\mathcal{T}$ usually yields the minimal genus surface.

Basic Fact: If M fibers over the circle then τ_M is monic, i.e. lead coefficient ± 1.

Current focus: For 15 crossing knots, does τ_M determine whether M fibers?

By Gabai can reduce to the case of *closed* manifolds.

Practical Trick: Proving that $N = M \setminus \Sigma$ is $\Sigma \times I$.

Start with a presentation for $\pi_1(N)$ coming from a triangulation, and then simplify that it using Tietze transformations. With luck (i.e. randomization), one gets a one-relator presentation of a surface group. This gives $N \cong \Sigma \times I$ by [Stallings 1960].

[Dunfield-Ramakrishnan 2008] Used this when $|\mathcal{T}| > 130$.

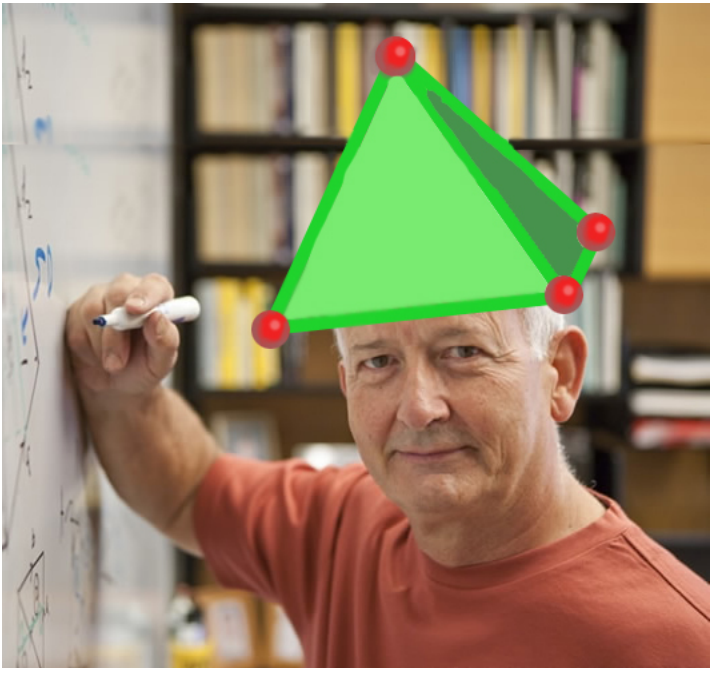
General approach uses Jaco-Rubinstein "crushing". Compare [Burton-Rubinstein-Tillmann 2009].

Future work: Considering τ_M as a function on the character variety.

Generic goals:

- Explain why genus bounds of τ_M are as good as those of Δ_M.
- Use ideal points associated to Seifert surfaces to show nonfibered implies τ_M is non-monic.
- Genus info?

Happy Birthday



Bus!